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DEVELOPMENT OF THE METHOD OF CHARACTERISTICS FOR A SYSTEM OF MAXWELL ELECTRODYNAMIC EQUATIONS

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Expressions for the velocities of propagation of the discontinuity surfaces in a homogeneous anisotropic medium have been obtained and equations of bicharacteristics have been derived. Using them, the surfaces of inverse velocities and three-dimensional wave fronts of electromagnetic waves have been constructed for different values of the permittivities and their features have been investigated.

Introduction. The first application of the classical method of characteristics to the Maxwell electrodynamics equations in the case of the isotropy and anisotropy of electrical properties was put forth by Levi-Civita in [1]. In this work, certain regularities of the propagation of electromagnetic waves were investigated using the equation of characteristics, and the equation of the wave surface was derived. In the works of Parton et al. [2–4], the method of characteristics was not developed for this direction in investigations, which is attributable to the cumbersomeness of the corresponding computations. However the capabilities and means of modern computer engineering make it possible to bypass these difficulties and to make a more comprehensive analysis of the characteristic equation for a system of electrodynamic equations.

Equation of Characteristics and the Surfaces of Inverse Velocities. Let us consider the following system of Maxwell equations for an anisotropic medium [1, 5]:

$$\begin{aligned} &\varepsilon_1\dot{E}_1 + \partial_3 H_2 - \partial_2 H_3 + ... = 0\,, \quad \mu\dot{H}_1 + \partial_2 E_3 - \partial_3 E_2 + ... = 0\,; \\ &\varepsilon_2\dot{E}_2 + \partial_1 H_3 - \partial_3 H_1 + ... = 0\,, \quad \mu\dot{H}_2 + \partial_3 E_1 - \partial_1 E_3 + ... = 0\,; \\ &\varepsilon_3\dot{E}_3 + \partial_2 H_1 - \partial_1 H_2 + ... = 0\,, \quad \mu\dot{H}_3 + \partial_1 E_2 - \partial_2 E_1 + ... = 0\,, \end{aligned} \qquad (1)$$

where $\partial_i = \dfrac{\partial}{\partial x_i}$, $i = \overline{1, 3}$; the dot denotes differentiation with respect to time.

Following [1, 5], we write the equation of the characteristic surface $Z(t, x_1, x_2, x_3) = 0$ for system (1) in the form

$$\begin{aligned} &p_0^2\,(p_0^4\,\mu^2\varepsilon_1\varepsilon_2\varepsilon_3 + p_0^2\,\mu\,(p_1^2\varepsilon_2\varepsilon_3 + p_2^2\varepsilon_1\varepsilon_3 + p_3^2\varepsilon_1\varepsilon_2 - \\ &- g^2\,(\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3)) + g^4\,(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) - \\ &- g^2\,(p_3^2\,(\varepsilon_1 + \varepsilon_2) + p_1^2\,(\varepsilon_2 + \varepsilon_3) + p_2^2\,(\varepsilon_1 + \varepsilon_3))) = 0\,. \end{aligned} \qquad (2)$$

Here

$$p_0 = \frac{\partial Z}{\partial t}\,, \quad p_i = \frac{\partial Z}{\partial x_i}\,, \quad i = \overline{1, 3}\,, \quad g^2 = p_1^2 + p_2^2 + p_3^2\,.$$

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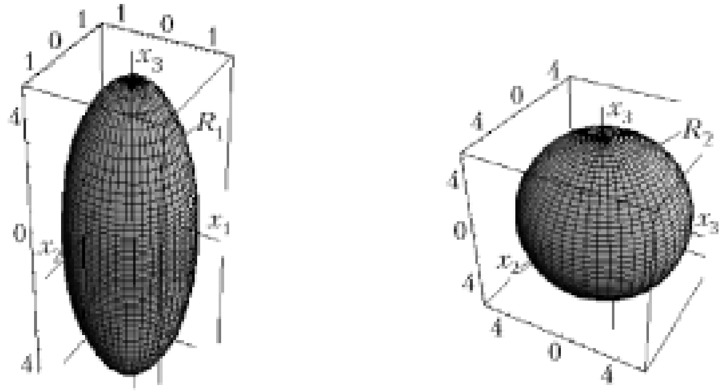

Fig. 1. Inverse-velocity surfaces R_1 and R_2, $\times 10^{-5}\sqrt{\mu}$, sec/m.

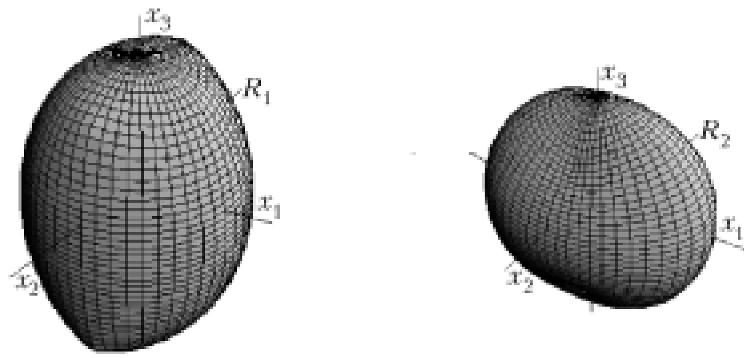

Fig. 2. Inverse-velocity surfaces R_1 and R_2 for the material whose permittivities satisfy the relations $\varepsilon_1/\varepsilon_3 = 2$ and $\varepsilon_3/\varepsilon_2 = 1/4$.

Equation (2) yields the existence of the stationary discontinuity surface $V = p_0/g = 0$ and of two electromagnetic waves whose velocities of propagation satisfy the following equation (velocity V is normal to the wave surface):

$$V^4\mu^2\varepsilon_1\varepsilon_2\varepsilon_3 + V^2\mu\,(\varepsilon_2\varepsilon_3\cos^2\alpha_1 + \varepsilon_1\varepsilon_3\cos^2\alpha_2 + \varepsilon_1\varepsilon_2\cos^2\alpha_3 -$$

$$-(\varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_3) + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 -$$

$$-(\cos^2\alpha_3\,(\varepsilon_1+\varepsilon_2) - \cos^2\alpha_1\,(\varepsilon_2+\varepsilon_3) - \cos^2\alpha_2\,(\varepsilon_1+\varepsilon_3)) = 0\,, \tag{3}$$

where $\cos\alpha_i = p_i/g$ are the direction cosines of the normal to the characteristic surface, $i = \overline{1, 3}$. From (3) we obtain

$$V_{1,2} = \frac{\sqrt{A \pm \sqrt{A^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3\,(\varepsilon_1\cos^2\alpha_1 + \varepsilon_2\cos^2\alpha_2 + \varepsilon_3\cos^2\alpha_3)}}}{\sqrt{2\mu\varepsilon_1\varepsilon_2\varepsilon_3}}\,,$$

$$A = \varepsilon_1\varepsilon_2\,(1-\cos^2\alpha_3) + \varepsilon_2\varepsilon_3\,(1-\cos^2\alpha_1) + \varepsilon_1\varepsilon_3\,(1-\cos^2\alpha_2)\,. \tag{4}$$

Figure 1 shows the surfaces of inverse velocities $R_{1,2} = 1/V_{1,2}$ constructed using formulas (4) for biaxial barium-niobate crystals of rhombic system ($\varepsilon_1 = 196{\cdot}10^{-11}$ F/m, $\varepsilon_2 = 201{\cdot}10^{-11}$ F/m, and $\varepsilon_3 = 28{\cdot}10^{-11}$ F/m [6]). We note that the inverse-velocity surfaces make it possible not only to find the velocity of propagation of the discontinuity surface but also to determine the direction of the transfer of electromagnetic energy.

Figure 1 yields that the inverse-velocity surfaces R_1 and R_2 are an ellipsoid and a sphere respectively. This is attributable to the fact that $\varepsilon_2/\varepsilon_1 \approx 1.026 \approx 1$, i.e., barium niobate is similar in electrical properties to semiconductor crystals of higher systems of symmetry (uniaxial crystals). If the values of the constants ε_1, ε_2, and ε_3 markedly differ, these surfaces of inverse velocities have quite a complex form for electromagnetic waves. Figure 2 shows the inverse-velocity surfaces R_2 for materials whose permittivities satisfy certain relations which are selected so as to show possible physical effects (the value of μ is shown arbitrarily).

The inverse-velocity surface R_2 is predominantly convex but there are four portions where the surfaces change their curvature and the convexity becomes concavity (Fig. 2). The appearance of such portions on the inverse-velocity surface points to the occurrence of four lacunas on the wave surface of an electromagnetic wave which propagates with a velocity V_2. The inverse-velocity surfaces R_1 do not possess the above characteristic properties, i.e., the propagation of electromagnetic waves with a velocity V_1 is not accompanied by the formation of lacunas. For materials characterized by other analogous permittivity ratios, the surfaces of inverse velocities have the same form. Thus, to obtain the inverse-velocity surfaces for the material with $\varepsilon_1/\varepsilon_3 = 1/2$ and $\varepsilon_3/\varepsilon_2 = 4$ one should rotate the surfaces R_1 and R_2 (presented in Fig. 2) by 90° about the x_1 axis; if $\varepsilon_1/\varepsilon_3 = \varepsilon_3/\varepsilon_2 = 2$, one should successively rotate R_1 and R_2 by 90° about the x_1 and x_3 axes, and so on. We note that the inverse-velocity surfaces have been constructed in the spherical coordinate system (r, φ, θ), where it was taken that $r = 1$, $\cos\alpha_1 = \sin\theta\cos\varphi$, $\cos\alpha_2 = \sin\theta\sin\varphi$, and $\cos\alpha_3 = \cos\theta$.

Bicharacteristics and Wave Surfaces. Let us express p_0 from Eq. (2):

$$p_0^{(1,2)} = \frac{\sqrt{B \pm \sqrt{B^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3 g^2\,(\varepsilon_1 p_1^2 + \varepsilon_2 p_2^2 + \varepsilon_3 p_3^2)}}}{\sqrt{2\mu\varepsilon_1\varepsilon_2\varepsilon_3}},$$

$$B = \varepsilon_1\varepsilon_2\,(p_1^2 + p_2^3) + \varepsilon_2\varepsilon_3\,(p_2^2 + p_3^2) + \varepsilon_1\varepsilon_3\,(p_1^2 + p_3^2)\,. \tag{5}$$

From (5), by differentiating $p_0^{(1,2)}$ with respect to the parameters p_i, $i = \overline{1, 3}$, we find the equations of bicharacteristics

$$\frac{\partial p_0^{(1,2)}}{\partial p_1} = \frac{dx_1^{(1,2)}}{dt} = p_1\varepsilon_1\,\frac{\varepsilon_2 + \varepsilon_3 \pm \dfrac{(\varepsilon_2 - \varepsilon_3)\,(p_1^2\varepsilon_1\,(\varepsilon_2 - \varepsilon_3) + p_2^2\varepsilon_2\,(\varepsilon_1 - \varepsilon_3) + p_3^2\varepsilon_3\,(\varepsilon_2 - \varepsilon_1))}{\sqrt{B^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3 g^2\,(\varepsilon_1 p_1^2 + \varepsilon_2 p_2^2 + \varepsilon_3 p_3^2)}}}{\sqrt{2\mu\varepsilon_1\varepsilon_2\varepsilon_3}\,\sqrt{B \pm \sqrt{B^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3 g^2\,(\varepsilon_1 p_1^2 + \varepsilon_2 p_2^2 + \varepsilon_3 p_3^2)}}},$$

$$\frac{\partial p_0^{(1,2)}}{\partial p_2} = \frac{dx_2^{(1,2)}}{dt} = p_2\varepsilon_2\,\frac{\varepsilon_1 + \varepsilon_3 \pm \dfrac{(\varepsilon_1 - \varepsilon_3)\,(p_1^2\varepsilon_1\,(\varepsilon_2 - \varepsilon_3) + p_2^2\varepsilon_2\,(\varepsilon_1 - \varepsilon_3) + p_3^2\varepsilon_3\,(\varepsilon_2 - \varepsilon_1))}{\sqrt{B^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3 g^2\,(\varepsilon_1 p_1^2 + \varepsilon_2 p_2^2 + \varepsilon_3 p_3^2)}}}{\sqrt{2\mu\varepsilon_1\varepsilon_2\varepsilon_3}\,\sqrt{B \pm \sqrt{B^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3 g^2\,(\varepsilon_1 p_1^2 + \varepsilon_2 p_2^2 + \varepsilon_3 p_3^2)}}}, \tag{6}$$

$$\frac{\partial p_0^{(1,2)}}{\partial p_3} = \frac{dx_3^{(1,2)}}{dt} = p_3\varepsilon_3\,\frac{\varepsilon_1 + \varepsilon_2 \pm \dfrac{(\varepsilon_1 - \varepsilon_2)\,(p_1^2\varepsilon_1\,(\varepsilon_3 - \varepsilon_2) + p_2^2\varepsilon_2\,(\varepsilon_1 - \varepsilon_3) + p_3^2\varepsilon_3\,(\varepsilon_1 - \varepsilon_2))}{\sqrt{B^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3 g^2\,(\varepsilon_1 p_1^2 + \varepsilon_2 p_2^2 + \varepsilon_3 p_3^2)}}}{\sqrt{2\mu\varepsilon_1\varepsilon_2\varepsilon_3}\,\sqrt{B \pm \sqrt{B^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3 g^2\,(\varepsilon_1 p_1^2 + \varepsilon_2 p_2^2 + \varepsilon_3 p_3^2)}}}.$$

Equation (6) yields that the bicharacteristics are functions of zero order and do not depend on the time t. Taking into account that $p_i = g\cos\alpha_i$, $i = \overline{1, 3}$, we obtain

$$x_1^{(1,2)} = \frac{\cos\alpha_1\varepsilon_1}{2\mu\varepsilon_1\varepsilon_2\varepsilon_3 V_{1,2}}\,(\varepsilon_2 + \varepsilon_3 \pm (\varepsilon_2 - \varepsilon_3)\;\frac{\varepsilon_1\,(\varepsilon_2 - \varepsilon_3)\cos^2\alpha_1 + \varepsilon_2\,(\varepsilon_1 - \varepsilon_3)\cos^2\alpha_2 + \varepsilon_3\,(\varepsilon_2 - \varepsilon_1)\cos^2\alpha_3}{\sqrt{A^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3\,(\varepsilon_1\cos^2\alpha_1 + \varepsilon_2\cos^2\alpha_2 + \varepsilon_3\cos^2\alpha_3)}}\;t\,,$$

$$x_2^{(1,2)} = \frac{\cos\alpha_2\varepsilon_2}{2\mu\varepsilon_1\varepsilon_2\varepsilon_3 V_{1,2}}\,(\varepsilon_1 + \varepsilon_3 \pm (\varepsilon_1 - \varepsilon_3) \times$$

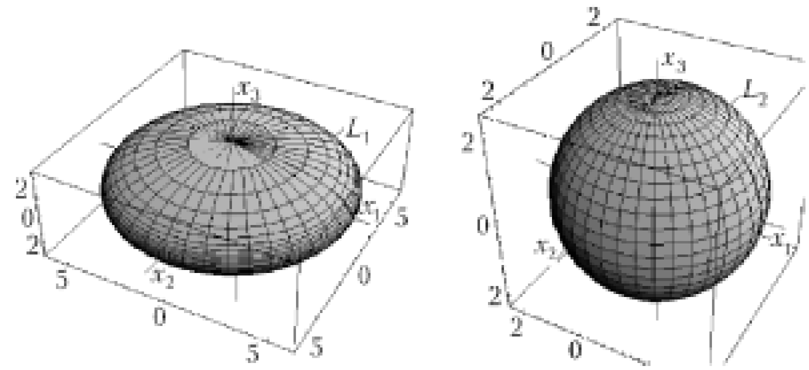

Fig. 3. Wave surfaces L_1 and L_2 of electromagnetic waves, $\times 10^{-5}\sqrt{\mu^{-1}}$, m.

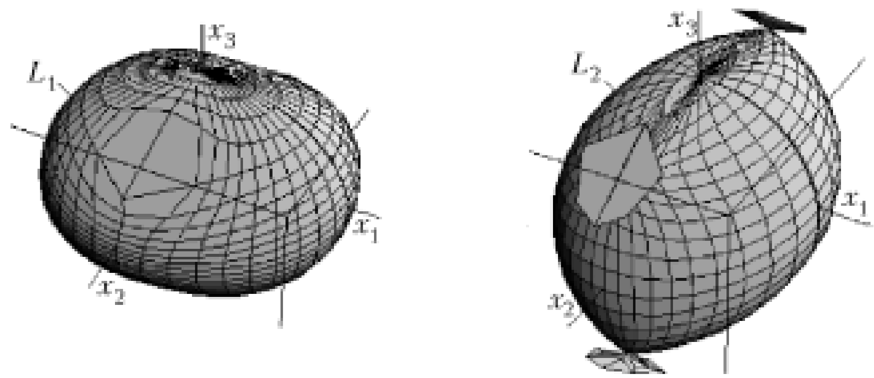

Fig. 4. Wave surfaces L_1 and L_2 of electromagnetic waves propagating in the material with permittivities of $\varepsilon_1/\varepsilon_3 = 2$ and $\varepsilon_3/\varepsilon_2 = 1/4$.

$$\times \frac{\varepsilon_1 (\varepsilon_2 - \varepsilon_3) \cos^2 \alpha_1 + \varepsilon_2 (\varepsilon_1 - \varepsilon_3) \cos^2 \alpha_2 + \varepsilon_3 (\varepsilon_2 - \varepsilon_1) \cos^2 \alpha_3}{\sqrt{A^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3 (\varepsilon_1 \cos^2 \alpha_1 + \varepsilon_2 \cos^2 \alpha_2 + \varepsilon_3 \cos^2 \alpha_3)}}\, t\,, \tag{7}$$

$$x_3^{(1,2)} = \frac{\cos \alpha_3 \varepsilon_3}{2\mu\varepsilon_1\varepsilon_2\varepsilon_3 V_{1,2}} (\varepsilon_1 + \varepsilon_2 \pm (\varepsilon_1 - \varepsilon_2) \times$$

$$\times \frac{\varepsilon_1 (\varepsilon_3 - \varepsilon_2) \cos^2 \alpha_1 + \varepsilon_2 (\varepsilon_1 - \varepsilon_3) \cos^2 \alpha_2 + \varepsilon_3 (\varepsilon_2 - \varepsilon_1) \cos^2 \alpha_3}{\sqrt{A^2 - 4\varepsilon_1\varepsilon_2\varepsilon_3 (\varepsilon_1 \cos^2 \alpha_1 + \varepsilon_2 \cos^2 \alpha_2 + \varepsilon_3 \cos^2 \alpha_3)}}\, t\,.$$

Formulas (7) make it possible to determine the coordinates ($x_1^{(1)}$, $x_2^{(1)}$, $x_3^{(1)}$) and ($x_1^{(2)}$, $x_2^{(2)}$, $x_3^{(2)}$) of the points of the medium reached by the wave surfaces L_1 and L_2 of the electromagnetic waves (propagating with velocities V_1 and V_2 respectively) at the instant of time t. Figure 3 gives the wave surfaces $L_{1,2}$ in barium niobate at the instant of time t = 1; these surfaces have been constructed using formulas (7) in the parametric coordinate system where $\cos \alpha_1 = \cos u \cos v$, $\cos \alpha_2 = \sin u \cos v$, and $\cos \alpha_3 = \sin v$. Both surfaces are lacuna-free and represent an ellipsoid (L_1) and a sphere (L_2), which is also attributable to the small difference between the permittivities ε_1 and ε_2.

Another form is assumed by the wave surfaces L_1 and L_2 of electromagnetic waves propagating in materials whose permittivities satisfy the relations $\varepsilon_1/\varepsilon_3 = 2$ and $\varepsilon_3/\varepsilon_2 = 1/4$ (Fig. 4).

As follows from Fig. 4, the surface L_2 contains four lacunas which have the form of cosines (the base of a lacuna is not a circle) and are located symmetrically relative to the coordinate planes (for the case of Fig. 4 the plane $x_1 = 0$ goes through the lacunas relative to the planes $x_2 = 0$ and $x_3 = 0$). The wave surface L_1 has no lacunas and represents an ellipsoid oblate in the direction of the bisectors of coordinate quarters of the plane $x_1 = 0$. The wave

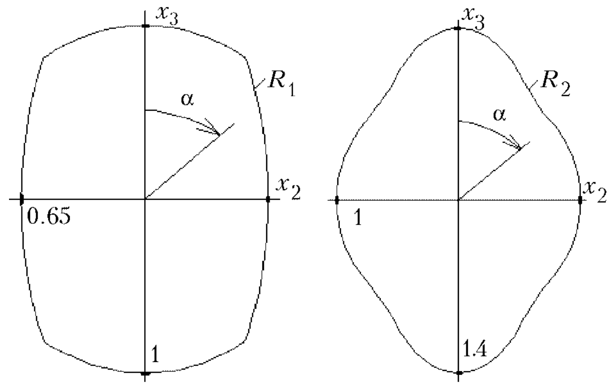

Fig. 5. Curves of the inverse velocities R_1 and R_2 in the coordinate plane x_1 = 0, sec/m.

surfaces for materials whose permittivities satisfy the relations $\varepsilon_1/\varepsilon_3$ = 1/2, $\varepsilon_3/\varepsilon_2$ = 4, $\varepsilon_1/\varepsilon_3 = \varepsilon_3/\varepsilon_2$ = 2, $\varepsilon_1/\varepsilon_3$ = 4, $\varepsilon_3/\varepsilon_2$ = 1/2, $\varepsilon_1/\varepsilon_3$ = 1/4, $\varepsilon_3/\varepsilon_2$ = 2, and $\varepsilon_1/\varepsilon_3 = \varepsilon_3/\varepsilon_2$ = 1/2 have an analogous form.

We note that the wave surfaces $L_{1,2}$ coincide in form with the surfaces of ray velocities (or the ray surfaces); the values of the ray velocities can be calculated using (6) from the formula

$$P_{1,2} = \sum_{k=1}^{3} \left(\frac{\partial p_0^{(1,2)}}{\partial p_k} \right).$$

Sections of the Inverse-Velocity Surfaces by Coordinate Planes. Let us consider the sections of the inverse-velocity surfaces $R_{1,2}$ and the wave surfaces $L_{1,2}$ by the coordinate planes x_i = 0, $i = \overline{1,3}$. Expressions (3) for the velocities of electromagnetic waves in the planes x_1 = 0 (cos α_1 = 0, cos α_2 = sin α, and cos α_3 = cos α), x_2 = 0 (cos α_1 = sin α, cos α_2 = 0, and cos α_3 = cos α), and x_3 = 0 (cos α_1 = sin α, cos α_2 = cos α, and cos α_3 = 0) assume the following form:

$$V_{1,2}^{(1)} = \frac{\sqrt{\varepsilon_2\varepsilon_3 + \varepsilon_1 a \pm |\varepsilon_2\varepsilon_3 - \varepsilon_1 a|}}{\sqrt{2\mu\varepsilon_1\varepsilon_2\varepsilon_3}}, \quad a = \varepsilon_3 \cos^2\alpha + \varepsilon_2 \sin^2\alpha ;$$

$$V_{1,2}^{(2)} = \frac{\sqrt{\varepsilon_1\varepsilon_3 + \varepsilon_2 b \pm |\varepsilon_1\varepsilon_3 - \varepsilon_2 b|}}{\sqrt{2\mu\varepsilon_1\varepsilon_2\varepsilon_3}}, \quad b = \varepsilon_3 \cos^2\alpha + \varepsilon_1 \sin^2\alpha ; \qquad (8)$$

$$V_{1,2}^{(3)} = \frac{\sqrt{\varepsilon_1\varepsilon_2 + \varepsilon_3 c \pm |\varepsilon_1\varepsilon_2 - \varepsilon_3 c|}}{\sqrt{2\mu\varepsilon_1\varepsilon_2\varepsilon_3}}, \quad c = \varepsilon_1 \cos^2\alpha + \varepsilon_2 \sin^2\alpha .$$

Here the subscript corresponds to the number of the coordinate plane. We apply (8) to construction of the curves of inverse velocities in the coordinate planes x_i = 0, $i = \overline{1,3}$, in the case where the permittivities of the material satisfy the relations $\varepsilon_1/\varepsilon_3$ = 2 and $\varepsilon_3/\varepsilon_2$ = 1/4 (Fig. 5).

In constructing the inverse-velocity curves, the value of the permeability μ is taken arbitrarily and the time t is selected so that the value of V_1 is 1 m/sec on the x_3 axis. The inverse-velocity curve $R_1 = 1/V_1$ represents an ellipsoid with semiaxes of 0.65 and 1 m/sec in the coordinate plane x_2 = 0 and a circle with a radius of 0.65 m/sec in the x_3 = 0 plane; the curve $R_2 = 1/V_2$ is a circle with a radius of 1.4 sec/m in the plane x_2 = 0 and an ellipse with semiaxes of 1 and 1.4 sec/m in the plane x_2 = 0.

The curve of R_2 has its characteristic properties which point to the formation of a lacuna (loop) on the curve of the wave front in the plane x_1 = 0 and lie in the existence of four portions of the curve with two inflection points on each. Another approach to the detection of such characteristic properties lies in finding two points of tangency of one tangent to the inverse-velocity curves. Both approaches have received wide acceptance in investigating the condi-

TABLE 1. Conditions of Occurrence of Lacunas on the Wave Surface L_2

Conditions	Plane		
	$x_1 = 0$	$x_2 = 0$	$x_3 = 0$
1	$\varepsilon_1/\varepsilon_3 \le 0.85$ $\varepsilon_3/\varepsilon_2 \ge 1.15$ $\varepsilon_2/\varepsilon_1 \le 0.85$	$\varepsilon_1/\varepsilon_3 \ge 1.15$ $\varepsilon_3/\varepsilon_2 \le 0.85$ $\varepsilon_2/\varepsilon_1 \le 0.85$	$\varepsilon_1/\varepsilon_3 \ge 1.15$ $\varepsilon_3/\varepsilon_2 \ge 1.15$ $\varepsilon_2/\varepsilon_1 \le 0.85$
2	$\varepsilon_1/\varepsilon_3 \ge 1.15$ $\varepsilon_3/\varepsilon_2 \le 0.85$ $\varepsilon_2/\varepsilon_1 \ge 1.15$	$\varepsilon_1/\varepsilon_3 \le 0.85$ $\varepsilon_3/\varepsilon_2 \ge 1.15$ $\varepsilon_2/\varepsilon_1 \ge 1.15$	$\varepsilon_1/\varepsilon_3 \le 0.85$ $\varepsilon_3/\varepsilon_2 \le 0.85$ $\varepsilon_2/\varepsilon_1 \ge 1.15$

tions of appearance of lacunas occurring in propagation of elastic waves in anisotropic media [7–9]. The analogous analysis of the curves of the inverse velocities of electromagnetic waves R_2 in three coordinate planes makes it possible to formulate two groups of occurrence conditions for lacunas in the form of pairs of inequalities which must be satisfied by the permittivities of the medium (see Table 1).

The conditions formulated in Table 1 are approximate in character, and the error of a numerical value on the right-hand side of the inequalities can attain ± 0.05. When at least one inequality of conditions 1 or 2 is violated, no lacunas occur on the wave front of the electromagnetic wave propagating with a velocity V_2. Thus, for barium niobate we have $\varepsilon_1/\varepsilon_3 = 7$, $\varepsilon_3/\varepsilon_2 = 0.14$, and $\varepsilon_2/\varepsilon_1 = 1.03$, i.e., the first condition of occurrence of lacunas in the plane $x_2 = 0$ is not fulfilled (or the second condition for the plane $x_1 = 0$).

Wave Fronts in Coordinate Planes. To construct the curves of the wave front $L_{1,2}$ of electromagnetic waves in coordinate planes we write, using (7), the expressions for the bicharacteristics in these planes (the subscript denotes the number of the coordinate axis, the superscript denotes the coordinate plane, the "+" sign relates to L_1, and the "–" sign relates to L_2):

$$x_2^{(1)} = \frac{\sqrt{\varepsilon_2}\,\sin\alpha\,((\varepsilon_1+\varepsilon_3)\,|\varepsilon_2\varepsilon_3-\varepsilon_1 a| \pm (\varepsilon_1-\varepsilon_3)\,(\varepsilon_1 a-\varepsilon_2\varepsilon_3))}{\sqrt{2\mu\varepsilon_1\varepsilon_3}\,|\varepsilon_2\varepsilon_3-\varepsilon_1 a|\,\sqrt{\varepsilon_2\varepsilon_3+\varepsilon_3 a-|\varepsilon_2\varepsilon_3-\varepsilon_1 a|}}\,t\,,$$

$$x_3^{(1)} = \frac{\sqrt{\varepsilon_3}\,\sin\alpha\,((\varepsilon_1+\varepsilon_2)\,|\varepsilon_2\varepsilon_3-\varepsilon_1 a| \pm (\varepsilon_1-\varepsilon_2)\,(\varepsilon_1 a-\varepsilon_2\varepsilon_3))}{\sqrt{2\mu\varepsilon_1\varepsilon_2}\,|\varepsilon_2\varepsilon_3-\varepsilon_1 a|\,\sqrt{\varepsilon_2\varepsilon_3+\varepsilon_3 a-|\varepsilon_2\varepsilon_3-\varepsilon_1 a|}}\,t\,; \tag{9}$$

$$x_1^{(2)} = \frac{\sqrt{\varepsilon_1}\,\sin\alpha\,((\varepsilon_2+\varepsilon_3)\,|\varepsilon_1\varepsilon_3-\varepsilon_2 b| \pm (\varepsilon_2-\varepsilon_3)\,(\varepsilon_2 b-\varepsilon_1\varepsilon_3))}{\sqrt{2\mu\varepsilon_2\varepsilon_3}\,|\varepsilon_1\varepsilon_3-\varepsilon_2 b|\,\sqrt{\varepsilon_1\varepsilon_3+\varepsilon_2 b-|\varepsilon_1\varepsilon_3-\varepsilon_2 b|}}\,t\,,$$

$$x_3^{(2)} = \frac{\sqrt{\varepsilon_3}\,\sin\alpha\,((\varepsilon_1+\varepsilon_2)\,|\varepsilon_1\varepsilon_3-\varepsilon_2 b| \pm (\varepsilon_2-\varepsilon_1)\,(\varepsilon_2 b-\varepsilon_1\varepsilon_3))}{\sqrt{2\mu\varepsilon_1\varepsilon_2}\,|\varepsilon_1\varepsilon_3-\varepsilon_2 b|\,\sqrt{\varepsilon_1\varepsilon_3+\varepsilon_2 b-|\varepsilon_1\varepsilon_3-\varepsilon_2 b|}}\,t\,; \tag{10}$$

$$x_1^{(3)} = \frac{\sqrt{\varepsilon_1}\,\sin\alpha\,((\varepsilon_2+\varepsilon_3)\,|\varepsilon_1\varepsilon_2-\varepsilon_3 c| \pm (\varepsilon_3-\varepsilon_2)\,(\varepsilon_3 c-\varepsilon_1\varepsilon_2))}{\sqrt{2\mu\varepsilon_2\varepsilon_3}\,|\varepsilon_1\varepsilon_2-\varepsilon_3 c|\,\sqrt{\varepsilon_1\varepsilon_2+\varepsilon_3 c-|\varepsilon_1\varepsilon_2-\varepsilon_3 c|}}\,t\,,$$

$$x_2^{(3)} = \frac{\sqrt{\varepsilon_2}\,\cos\alpha\,((\varepsilon_1+\varepsilon_3)\,|\varepsilon_1\varepsilon_2-\varepsilon_3 c| \pm (\varepsilon_3-\varepsilon_1)\,(\varepsilon_3 c-\varepsilon_1\varepsilon_2))}{\sqrt{2\mu\varepsilon_1\varepsilon_3}\,|\varepsilon_1\varepsilon_2-\varepsilon_3 c|\,\sqrt{\varepsilon_1\varepsilon_2+\varepsilon_3 c-|\varepsilon_1\varepsilon_2-\varepsilon_3 c|}}\,t\,. \tag{11}$$

The constructions of the wave fronts in the coordinate planes $x_2 = 0$ and $x_3 = 0$ carried out using (10) and (11) show that for materials with permittivities satisfying the second condition in the plane $x_1 = 0$ (see Table 1) the

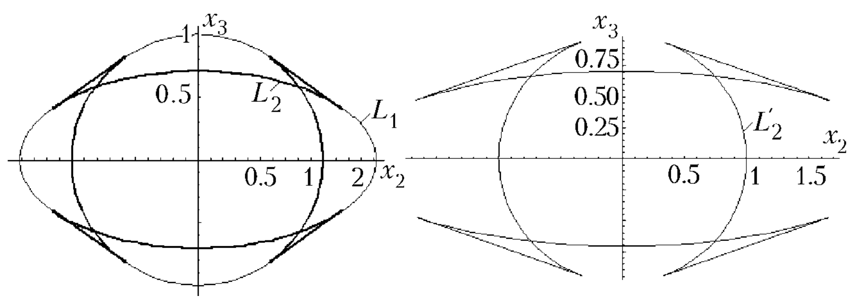

Fig. 6. Sections of the wave fronts of electromagnetic waves by the plane x_1 = 0 in materials with different permittivity ratios: $L_{1,2}$, $\varepsilon_1/\varepsilon_3$ = 2 and $\varepsilon_3/\varepsilon_2$ = 1/4 m; L'_2, $\varepsilon_1/\varepsilon_3$ = 5 and $\varepsilon_3/\varepsilon_2$ = 1/10 m.

wave front is a circle in the plane $x_2 = 0$ and an ellipse in the plane $x_3 = 0$. Figure 6 shows the sections of the wave surfaces L_1 and L_2 of electromagnetic waves by the plane $x_1 = 0$ for anisotropic media with different permittivity ratios.

In constructing the wave fronts L_2, the value of the permeability and the time t are selected so that the point of intersection of the wave front and the axis x_2 is at a distance of 1 m from the disturbance source.

From Fig. 6 it follows that the wave fronts L_1 and L_2 superimpose on one another at the sites of location of the lacunas; the front L_2 does not go beyond the scope of the front L_1. We note that the form of the lacunas markedly changes when the ratios $\varepsilon_1/\varepsilon_2$ and $\varepsilon_3/\varepsilon_2$ deviate from the numerical values on the right-hand side of the inequalities (see Table 1). Thus, on the L'_2 curve the edges of the lacunas are nonsymmetric relative to the axis going through the origin of coordinates and the point at which the branches of the wave front intersect (see Fig. 6).

Thus, the wave surfaces of the electromagnetic waves in biaxial crystals ($\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3$) are either ellipsoids or represent surfaces similar to those shown in Fig. 4 in the case of fulfillment of the conditions (see Table 1). In uniaxial crystals, the surfaces $L_{1,2}$ are an ellipsoid and a sphere respectively, while in cubic crystals they are two spheres.

The most widespread approach to investigation of the equation of characteristics in the theory of discontinuous solutions and the dispersion equations in the theory of plane waves is the solution of them in the coordinate planes $x_i = 0$, $i = \overline{1, 3}$. Thus, for example, in our case in the plane $x_1 = 0$ ($\cos \alpha_1 = 0$) we obtain, from (3), the following equation:

$$V^2 (\mu\varepsilon_1 V^2 - 1) (\varepsilon_2 \cos^2 \alpha_2 + \varepsilon_3 \cos^2 \alpha_3 - \mu\varepsilon_2\varepsilon_3 V^2) = 0 .$$

This yields the existence of the stationary discontinuity surface $V = 0$ and of two electromagnetic waves propagating with velocities

$$V_1 = 1/\sqrt{\mu\varepsilon_1} , \quad V_2 = \sqrt{(\varepsilon_2 \cos^2 \alpha_2 + \varepsilon_3 \cos^2 \alpha_3)/\mu\varepsilon_2\varepsilon_3} . \tag{12}$$

The inverse-velocity curves constructed using (12) represent a circle and an ellipse for any relation between the permittivities and they do not point to the appearance of any features in propagation of electromagnetic waves.

Let us multiply the left- and right-hand sides of (12) by g and take into account that $p_0 = V_g$ and $p_i = g \cos \alpha_i$, $i = \overline{1, 3}$. We obtain

$$p_0^{(1)} = g/\sqrt{\mu\varepsilon_1} , \quad p_0^{(2)} = \sqrt{(\varepsilon_2 p_2^2 + \varepsilon_3 p_3^2)/\mu\varepsilon_2\varepsilon_3} . \tag{13}$$

We find the bicharacteristics forming the wave fronts of the electromagnetic waves in the plane $x_1 = 0$:

$$\frac{\partial p_0^{(1)}}{\partial p_k} = \frac{dx_k^{(1)}}{dt} = \frac{p_k}{g \sqrt{\mu\varepsilon_1}} , \quad \frac{\partial p_0^{(2)}}{\partial p_k} = \frac{dx_k^{(2)}}{dt} = \frac{\varepsilon_k p_k}{\sqrt{\mu\varepsilon_2\varepsilon_3 (\varepsilon_2 p_2^2 + \varepsilon_3 p_3^2)}} \tag{14}$$

or

$$x_k^{(1)} = t\cos\alpha_k/\sqrt{\mu\varepsilon_1}\ ,\quad x_k^{(2)} = \frac{t\varepsilon_k\cos\alpha_k}{\sqrt{\mu\varepsilon_2\varepsilon_3\,(\varepsilon_2\cos^2\alpha_2 + \varepsilon_3\cos^2\alpha_3)}}\ ,\quad k = 2, 3\ . \qquad (15)$$

Upon simple manipulations, from (15) we will have

$$x_2^2 + x_3^2 = 1/\sqrt{\mu\varepsilon_1}\ ,\quad x_2^2/\varepsilon_2 + x_3^2/\varepsilon_3 = 1/\sqrt{\mu\varepsilon_2\varepsilon_3}\ ,$$

i.e., the wave fronts in the plane $x_1 = 0$ represent either circles or ellipses and contain lacunas for none of the values of ε_1, ε_2, and ε_3.

Using (14) we find the ray velocity of propagation of the electromagnetic waves $P_{1,2}$:

$$P_1 = 1/\sqrt{\mu\varepsilon_1}\ ,\quad P_2 = \frac{\sqrt{\varepsilon_2^2\cos^2\alpha_2 + \varepsilon_3^2\cos^2\alpha_3)}}{\sqrt{\mu\varepsilon_2\varepsilon_3\,(\varepsilon_2\cos^2\alpha_2 + \varepsilon_3\cos^2\alpha_3)}}\ . \qquad (16)$$

The values of the ray and phase velocities of propagation of the electromagnetic waves determined by formulas (16) and (12) coincide for the corresponding values of the permittivities and the permeabilities and the angles of incidence of the normal to the wave surface. It is easily seen that analogous results are yielded by solution of Eq. (3) in the coordinate planes $x_2 = 0$ and $x_3 = 0$.

CONCLUSIONS

The results of solution of the equation of propagation of the discontinuity surface in coordinate planes partially coincide with those obtained in solving the equation of characteristics in the general case; therefore, one should carry out further investigations, relying on the physicomechanical properties of actual media and the regularities of propagation of electromagnetic waves in anisotropic media established experimentally.

NOTATION

E = (E_1, E_2, E_3) and **H** = (H_1, H_2, H_3), electric and magnetic field strengths; μ, permeability of the medium; ε_1, ε_2, and ε_3, dielectric constants of the anisotropic medium.

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